Msc (Sem-03) Paper-11
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Generalized Hahn-Banach theorem on real linear spaces and Complex linear space.

Statement:- Let L be a real linear space not necessarily normed and let p be a sublinear function on L i.e. p is a map from L into R satisfying
$$p(x+y) \le p(x)+p(y) \text{ for all } x \in L \text{ and all } \alpha \ge 0 \quad —(1)$$
$$p(\alpha x) = \alpha p(x) \text{ for all } x \in L \text{ and all } \alpha > 0 \quad —(2)$$
If f is a real linear functional defined on a linear subspace M such that $f(x) \le p(x)$ for all x in M, then there exists a real linear function F defined on the whole L such that $f = F$ on M and $F(x) \le p(x)$ for all $x \in N$.

If L is complex linear space, then condition (1) is the same for but (2) is modified to $p(\alpha x) = |\alpha| p(x)$ for all $x \in L$ and all scalars α. ~~And~~ If f is a complex linear functional on M such that $|f(x)| \le p(x)$ for all $x \in M$.

The conclusion in this case is the same except that we have
$$|F(x)| \le p(x) \ \forall x \in L.$$

Proof:- I. Let L be a real linear space.

If $x_0 \notin M$. Consider the subspace
$$M_0 = (M \cup \{x_0\}) = \{x + \alpha x_0 : x \in M, \alpha \text{ real}\}$$
spanned by M and x_0.

Define f_0 on M_0 by $f_0(x+\alpha x_0) = f(x+\alpha x_0)$ where r_0 is any real number so that f_0 is real valued. It is easy to see that f_0 is linear on M_0 and $f_0 = f$ on M. If x_1, x_2 are any vectors in M, then $f(x_0) - f(x_1) = f(x_2 - x_1) \le p(x_2 - x_1)$ by hypothesis
$$= p((x_2+x_0) - (x_1+x_0)) \le p(x_2+x_0) + p(-x_1-x_2) \quad \text{— by (1)}$$
so that $-f(x_1) - p(-x-x_0) \le -f(x_2) + p(x_2+x_0)$
Since this inequality holds for arbitrary $x_1, x_2 \in$
We conclude that

$$\sup_{y \in M} \{-f(y) - P(-y-x_0)\} \le r_0 \le \inf\{-f(y) + P(y+x_0)\}$$

Choose r_0 to be any real number such that

$$\sup\{-f(y) - P(-y-x_0)\} \le r_0 \le \inf_{y \in M}\{-f(y)+P(y+x_0)\}.$$

It follows that

$$-f(y) - P(-y-x_0) \le r_0 \le \{-f(y) + P(y+x_0)\} \quad \text{---(3)}$$

for all $y \in M$. With this choice of r_0, we shall show that $f_0(x) \le P(x)$ for all $x \in M_0$.

Let $w = x + \alpha x_0$ be an arbitrary element in M_0. If $\alpha = 0$, then $f_0(w) = f(x) \le P(x)$.

So let $\alpha \ne 0$ and put $y = x/\alpha$ in (3) to obtain

$$-f(x/\alpha) - P\left(-\frac{x}{\alpha} - x_0\right) \le r_0 \le -f(x/\alpha) + P(x/\alpha + x_0) \quad \text{---(4)}$$

for all $x \in M$. If $\alpha > 0$, then right hand inequality in (4) gives

$$r_0 \le -\frac{1}{\alpha} f(x) + \frac{1}{\alpha} P(x + \alpha x_0)$$

$$\Rightarrow f(x) + \alpha r_0 \le P(x + \alpha x_0) \Rightarrow f_0(x + \alpha x_0) \le P(x + \alpha x_0)$$

And if $\alpha < 0$, then the left hand inequality in (4) gives

$$r_0 \ge -f\left(\frac{x}{\alpha}\right) - P\left(-\frac{x}{\alpha} - x_0\right) = -f\left(\frac{x}{\alpha}\right) - P\left(-\frac{1}{\alpha}(x+\alpha x_0)\right)$$

$$= -\frac{1}{\alpha} f(x) - \left(-\frac{1}{\alpha}\right) P(x + \alpha x_0) \text{ by (1) since } -\frac{1}{\alpha} > 0$$

$$= -\frac{1}{\alpha} f(x) + \frac{1}{\alpha} P(x + \alpha x_0)$$

We now multiply both sides of this inequality by α. Since $\alpha < 0$, the inequality will be reversed.

$$\therefore \alpha r_0 + f(x) \le P(x + \alpha x_0) \Rightarrow f_0(x + \alpha x_0) \le P(x + \alpha x_0)$$

Thus when $\alpha \ne 0$, we obtain

$$f_0(x + \alpha x_0) \le P(x + \alpha x_0) \text{ for all } x \in M.$$

i.e. $f_0(x) \le P(w)$ for all $w \in M_0$. Thus f_0 is a real linear functional on M_0 such that $f_0(x) = f(x)$ for all $x \in M$ and $f_0(w) \le P(w)$ for all $w \in M_0$.

If $M_0 = L$, then we finish. If not we may repeat the process of extension but what guarantee is there that we shall ever extend to the whole space L.

It is at this print that zorn's lemma is needed. let P denote the set of all ordered prirs (f_λ, M_λ) where f_λ is an extension of f to the subspace $M_\lambda \supset M$

and $f_\lambda(x) \leq P(x)$ for all $x \in M_\lambda$.

Partially order P by setting $(f_\lambda, M_\lambda) \leq (f_\mu, M_\mu)$ iff $M_\lambda \subset M_\mu$ and $f_\lambda = f_\mu$ on M_λ. P is evidently non-emplty. Let $\varphi = \{f_i, M_i\}$ be a chain (i.e. a totally ordered set) in P. Then it is easy to see that φ has an upper bound

$$(\varphi, \cup M_i) \text{ where } \varphi(x) = f_i(x) \text{ for all } x \in M_i.$$

The print to be noted is $\cup M_i$ is a subspace of N and that φ is well-defined because of total ordering on φ.

Hence by zorn's lemma, P contains a maximal element (F, H). To complete the proof, we must show that $H = N$ Suppose, if possible, N contains H property. Then there exists $x_0 \in N - H$ and by first part of the theorem, F can be extended to a functional F_0 on $H_0 = (H \cup \{x_0\})$ which contains H property. But this contradicts the maximality of (F, H). Consequently we must have $H = N$ and the proof is complete.

II. Let L be a complex linear space.

Here f is a complex linear functional on M such that $|f(x)| \leq P(x)$ for all $x \in M$.

Let $f_1 = \mathrm{Re} f$, then $f_1(x) \leq |f(x)| \leq P$ and so by case II, f_1 can be extended to a linear map F_1 of L into R such that $F_1 = f_1$ on M and $F_1(x) \leq P(x)$ for all $x \in L$.

Define F by $F(x) = F_1(x) - iF_1(ix)$, $x \in L$.

Then it is easy to see that F is a linear functional on L such that $F = f$ on M. What remains to prove is that $|F(x)| \leq P(x)$ for all $x \in L$. Let $x \in L$ be arbitrary and write $F(x) = re^{i\theta}$ where $r \geq 0$ and θ is real. Then

$|F(x)| = r = e^{-i\theta} \cdot re^{i\theta}\, F(x) = F(e^{-i\theta}x) = F_1(e^{-i\theta}x)$ $\because r$ is real.

$\leq P(e^{-i\theta}) P(x)$ by (2) $= P(x)$.

Proved.

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